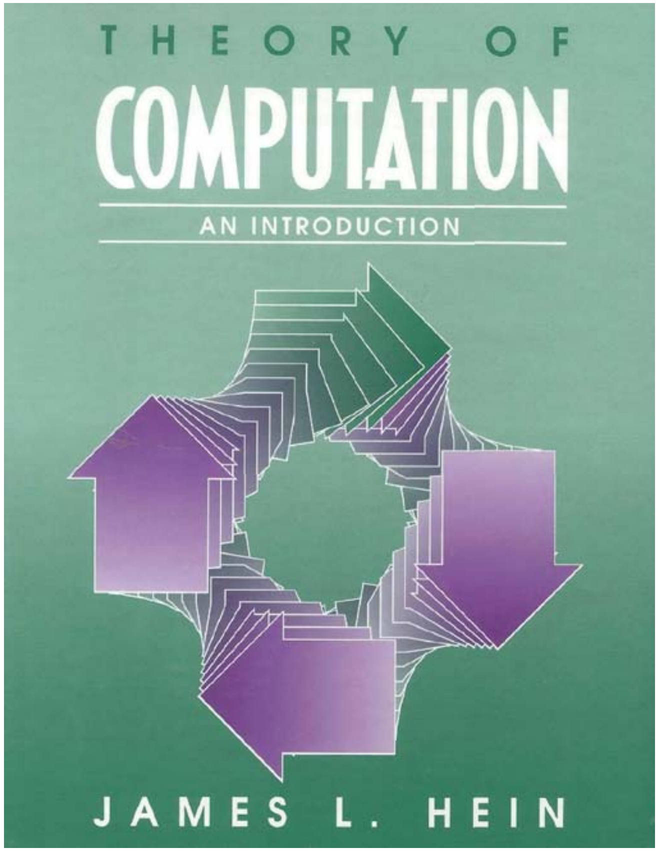
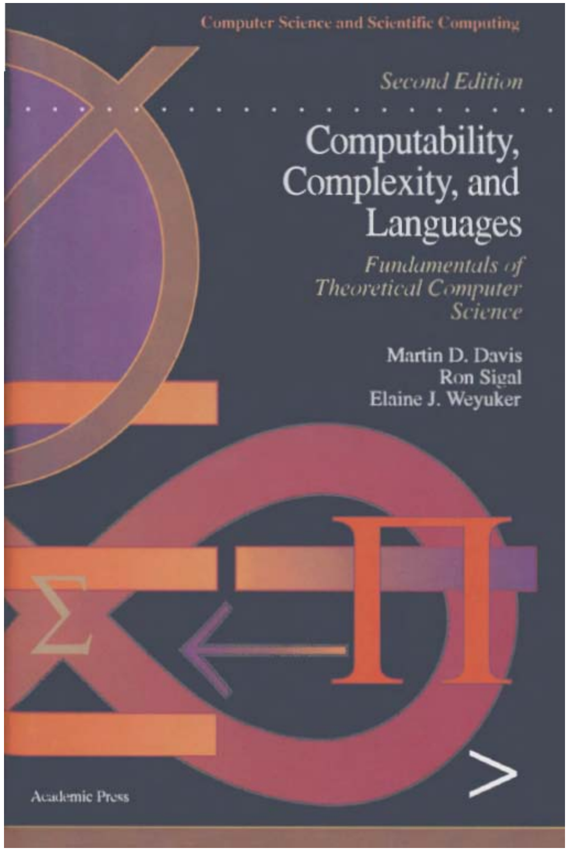
Theory of Computation

Lecture 05

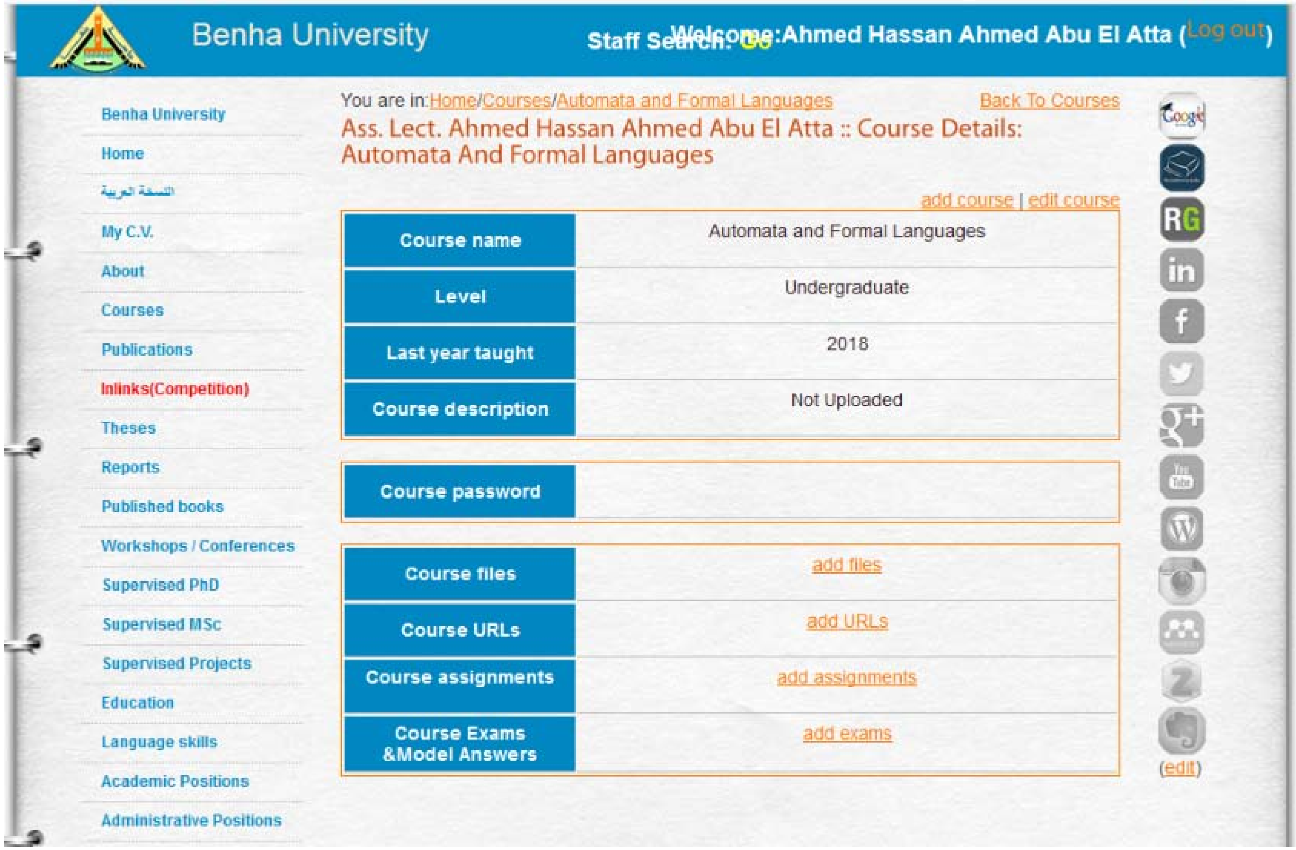
Books


THEORY OF
COMPUTATION
AN INTRODUCTION
JAMES L. HEIN


Computer Science and Scientific Computing
Second Edition
Computability, Complexity, and Languages
Fundamentals of Theoretical Computer Science
Martin D. Davis
Ron Sigal
Elaine J. Weyuker
Academic Press

PowerPoint

http://www.bu.edu.eg/staff/ahmedaboalatah14-courses/14767



Programs and Computable Functions

SIMPLE LANGUAGE III

Agenda

- Pairing Functions
- Examples
- Gödel Number
- Examples
- Coding Programs by Numbers
- Examples

Pairing Functions

The Pairing Functions is for coding pairs of numbers by single numbers.

$$\langle x, y\rangle = 2^x(2y + 1) - 1$$

If z is any given number, there is a unique solution x, y to the equation

$$\langle x, y\rangle = z$$

x *is the largest number such that* $2^x \mid (z + 1)$*, and y is then the* solution of the equation

$$2y + 1 = (z + 1)/2^x$$

Example <x, y>=??

<2, 3>

=

<0, 5>

=

<3, 2>

=

<4, 0>

=

Example <x, y>=??

<2, 3>

= $2^2(2*3+1)-1$

= 4(7)-1 = 27

<0, 5>

= $2^0(2*5+1)-1$

= 1(11)-1 = 10

<3, 2>

= $2^3(2*2+1)-1$

= 8(5)-1 = 39

<4, 0>

= $2^4(2*0+1)-1$

= 16(1)-1 = 15

Example z = ??

5

$= 2^x(2y + 1) - 1$

100

$= 2^x(2y + 1) - 1$

Example z = ??

5

$= 2^x(2y + 1) - 1$

$6 = 2^x(2y + 1)$

$2^1 * 3 = 2^x(2y + 1)$

$x = 1,$

$2y+1 = 3$

$y = 1$

$5 = <1, 1>$

100

$= 2^x(2y + 1) - 1$

$101 = 2^x(2y + 1)$

$2^0 * 101 = 2^x(2y + 1)$

$x = 0,$

$2y+1 = 101$

$y = 50$

$100 = <0, 50>$

Example z = ??

11

$= 2^x(2y + 1) – 1$

22

$= 2^x(2y + 1) – 1$

Example z = ??

11

$= 2^x(2y + 1) - 1$

$12 = 2^x(2y + 1)$

$2^2*3 = 2^x(2y + 1)$

$x = 2,$

$2y+1 = 3$

$y = 1$

$11 = <2, 1>$

22

$= 2^x(2y + 1) - 1$

$23 = 2^x(2y + 1)$

$2^0*23 = 2^x(2y + 1)$

$x = 0,$

$2y+1 = 23$

$y = 11$

$22 = <0, 11>$

Gödel Number

We define the Gödel number of the sequence (a_1 , … , a_n) to be the number

$$[a_1, \ldots, a_n] = \prod_{i=1}^{n} p_i^{a_i}$$

Example [a, b, c,…] = ??

[2, 3] =

[0, 2, 3] =

[2, 3, 0, 0, 0] =

Example [a, b, c,…] = ??

[2, 3] $= 2^2 * 3^3$

$= 4*27 = 108$

[0, 2, 3] $= 2^0 * 3^2 * 5^3$

$= 1*9*125 = 1125$

[2, 3, 0, 0, 0] $= 2^2 * 3^3 * 5^0 * 7^0 * 11^0$

$= 4*27 * 1*1*1 = 108$

Example [??, ??, ??] = P

30 = [ ]

120 = [ ]

Example [??, ??, ??] = P

$30 = 2^1*3^1*5^1$

$= [1, 1, 1]$

$120 = 2^3*3^1*5^1$

$= [3, 1, 1]$

Example [??, ??, ??] = P

39 =

Example [??, ??, ??] = P

39 = 3*13

$= 2^0*3^1*5^0*7^0*11^0*13^1$

= [0, 1, 0, 0, 0, 1]

Coding Programs by Numbers

We are going to associate with each program $\mathscr{P}$ of the language $\mathscr{S}$ a number, which we write $\#(\mathscr{P})$, in such a way that the program can be retrieved from its number. To begin with we arrange the variables in order as follows:

$$Y \; X_1 \;\; Z_1 \;\; X_2 \;\; Z_2 \;\; X_3 \;\; Z_3 \ldots .$$

Coding Programs by Numbers

Next we do the same for the labels:

$$A_1 \; B_1 \; C_1 \; D_1 \; E_1 \; A_2 \; B_2 \; C_2 \; D_2 \; E_2 \; A_3 \ldots .$$

We write $\#(V)$, $\#(L)$ for the position of a given variable or label in the appropriate ordering. Thus $\#(X_2) = 4$, $\#(Z_1) = \#(Z) = 3$, $\#(E) = 5$, $\#(B_2) = 7$.

Coding Programs by Numbers

Now let I be an instruction (labeled or unlabeled) of the language $\mathscr{S}$. Then we write

$$\#(I) = \langle a, \langle b, c \rangle \rangle$$

where

1. if I is unlabeled, then $a = 0$; if I is labeled L, then $a = \#(L)$;
2. if the variable V is mentioned in I, then $c = \#(V) - 1$;
3. if the statement in I is

$$V \leftarrow V \qquad \text{or} \qquad V \leftarrow V + 1 \qquad \text{or} \qquad V \leftarrow V - 1,$$

   then $b = 0$ or 1 or 2, respectively;
4. if the statement in I is

$$\text{IF } V \neq 0 \text{ GOTO } L'$$

   then $b = \#(L') + 2$.

Coding Programs by Numbers

Finally, let a program $\mathscr{P}$ consist of the instructions $I_1, I_2, \ldots, I_k$. Then we set

$$\#(\mathscr{P}) = [\#(I_1), \#(I_2), \ldots, \#(I_k)] - 1$$

Example

$X \leftarrow X + 1$

$[A] \quad X \leftarrow X + 1$

Example

$$X \leftarrow X + 1$$

$$[A] \quad X \leftarrow X + 1$$

$$\langle 0, \langle 1, 1 \rangle \rangle = \langle 0, 5 \rangle = 10$$

$$\langle 1, \langle 1, 1 \rangle \rangle = \langle 1, 5 \rangle = 21$$

$$[A] \quad X \leftarrow X + 1$$
$$\text{IF } X \neq 0 \text{ GOTO } A$$

$[A] \quad X \leftarrow X + 1$ $\qquad$ IF $X \neq 0$ GOTO A

$$[A] \quad X \leftarrow X + 1$$
$$\text{IF } X \neq 0 \text{ GOTO } A$$

$[A] \quad X \leftarrow X + 1$ $\qquad$ IF $X \neq 0$ GOTO A

$\langle 1, \langle 1, 1 \rangle \rangle = \langle 1, 5 \rangle = 21$ $\qquad$ $\#(I_2) = \langle 0, \langle 3, 1 \rangle \rangle = \langle 0, 23 \rangle = 46$

the number of this short program is

$$2^{21} \cdot 3^{46} - 1.$$

Determine the program whose number is 199

Determine the program whose number is 199

$$199 + 1 = 200 = 2^3 \cdot 3^0 \cdot 5^2 = [3, 0, 2].$$

Thus, if $\#(\mathscr{P}) = 199$, $\mathscr{P}$ consists of 3 instructions, the second of which is the unlabeled statement $Y \leftarrow Y$. We have

$$3 = \langle 2, 0 \rangle = \langle 2, \langle 0, 0 \rangle \rangle$$

and

$$2 = \langle 0, 1 \rangle = \langle 0, \langle 1, 0 \rangle \rangle.$$

Thus, the program is

$$\begin{aligned} [B]\, Y &\leftarrow Y \\ Y &\leftarrow Y \\ Y &\leftarrow Y + 1 \end{aligned}$$